DCT-003-1161001

DCT-003-1161001

Seat No. ______

M. Sc. (Sem. I) Examination

August - 2022

Mathematics : CMT-1001

(Algebra - I)

Faculty Code : 003
Subject Code : 1161001

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions : (1) Attempt any five questions from the following.
(2) There are total ten questions.
(3) Each question carries equal marks.

1 Answer following seven questions : **7×2=14**

(i) Define a normal subgroup of a group G and write down a proper normal subgroups of S_3, where $S_3 = \{e, \sigma, \sigma^2, \psi, \sigma\psi, \sigma^2\psi\}$.

(ii) In standard notation, prove or disprove that, S_3 is an abelian group.

(iii) Let $S_3 = \{e, \sigma, \sigma^2, \phi, \sigma\phi, \sigma^2\phi\}$. Take $K = \{e, \phi\}$. Write down all the left cosets of K in S_3.

(iv) Define internal direct product of a group G by its finite normal subgroups.

(v) Define term: Field. Give an example of a field.

(vi) Define terms: Prime element and irreducible element.

(vii) Define Euclidean Domain.

2 Answer following seven questions : **7×2=14**

(i) Let $\phi : G \to G'$ be a group homomorphism. Define ker ϕ and prove that, it is a subgroup of G.

(ii) Write down all the normal subgroups of A_4, where A_4 = {e, (1, 2) (3, 4), (1, 3) (2, 4), (1, 4) (2, 3), (1, 2, 3), (1, 3, 2), (1, 2, 4), (1, 4, 2), (1, 3, 4), (1, 4, 3), (2, 3, 4), (2, 4, 3)}.

(iii) Prove that, for any $m \in \mathbb{N}$, $m\mathbb{Z}$ is an ideal of $\mathbb{Z}$.

(iv) Prove or disprove, A_4 has no subgroup of order six.

(v) Let G be a group and $a \in G$. Prove that, $N(a) = \{g \in G / ga = ag\}$ is a subgroup of G.

(vi) Define term: External direct product of groups.

(vii) Define ring homomorphism and give two ring homomorphisms on the rings Z into Z.

3 Answer following two questions : **2×7=14**

(1) Let G be finite group. Let $O(G) = p^n$, for some prime p and $\in N$. Prove that, $O(Z(G)) > 1$. i.e. $Z(G) \neq \{e\}$.

(2) Let G be a group and N_i are normal subgroups of G, for all i = 1, 2, ..., k. Prove that, G is internal direct product of its subgroups N_1, N_2, ..., N_k if and only if it satisfying followings :

(i) $G = N_1 \cdot N_2 \cdot \ldots\ldots \cdot N_K$ and

(ii) $N_i \cap \prod_{i \neq j,\, j=1}^{k} N_j = \{e\}$, for all i = 1, 2, ..., k.

4 Answer following two questions : **2×7=14**

(a) State and Prove, First Isomorphism Theorem of Rings.

(b) State and Prove, First Sylow's Theorem.

5 Answer following two questions : **2×7=14**

(a) State and Prove, Third Isomorphism Theorem of Rings.

(b) State and Prove, Second Isomorphism Theorem of Groups.

6 Answer following two questions : **2×7=14**

(a) Let G be a group and H be a subgroup of G. Suppose $O(H) = \frac{1}{2} O(G)$. Prove that, H is a maximal normal subgroup of G.

(b) Prove or disprove, the center of a group G is a normal subgroup of G. Also prove that, G is an abelian group if and only if its center is itself.

7 Answer following two questions : **2×7=14**

(1) Let G be the internal direct product of its normal subgroups N_1, N_2, ..., N_k. Prove that, $N_1 \cdot N_2 \cdot \ldots \cdot N_k \cong N_1 \times N_2 \times \ldots \times N_k$, as groups.

(2) Let G be a finite group and $O(G) = pq$. Let $p < q$, p, q both are primes and p does not divide to $q - 1$. Prove that, G is a cyclic group.

8 Answer following two questions : **2×7=14**

(a) Let $G = \langle g \rangle$ be a cyclic group and $O(G) = mn$, where m and n are relatively primes. Let $H = \langle g^m \rangle$ and $K = \langle g^n \rangle$. Prove that, G is the internal direct product of its subgroups H and K.

(b) Let G be a finite group and p is divisor of $O(G)$, for some prime p. Let P be a sylow p-subgroup of G. Prove that, P is only Sylow p-subgroup of G if and only if P is the normal subgroup of G.

9 Answer following two questions : **2×7=14**

(1) Let $f : \mathbb{Q} \to \mathbb{Q}$ be a non-zero ring homomorphism. Prove that, f is the identity map on $\mathbb{Q}$.

(2) Let R be a commutative ring and P be a prime ideal of R. Prove that, P is a prime ideal of R if and only if, whenever $a, b \in R$ be two elements and $ab \in P$, then either $a \in P$ or $b \in P$.

10 Answer following one question : **1×14=14**

Let R be a ring and $1 \in R$. Let M be an ideal of R with M # R. Prove that, following statement are equivalent.

(a) M is a maximal ideal of R.

(b) R/M has no non-trivial ideal.

(c) $M + (x) = R$, for every $x \in R - M$.